Math. Zeitschr. 106, 343–354 (1968)

On the Theory of Soluble Lie Algebras

DONALD W. BARNES and HUMPHREY M. GASTINEAU-HILLS

In this paper, we develop for finite-dimensional soluble Lie algebras, a theory of saturated homomorphs and formations and of functors analogous to the theories developed for finite soluble groups by Gaschütz, Lubeseder, Schunck, Barnes and Kegel. Much of this requires little more than verbal alteration of the group theory form, replacing the words "*group*" and "*normal subgroup*" by "*algebra*" and "*ideal*" wherever they occur. Even where this translation is completely routine, for the sake of completeness, we give at least brief indications of the proofs.

All algebras in the following are finite-dimensional soluble Lie algebras over some fixed ground field F. Except where specifically stated, no assumptions are made about the field F. We write $A \subseteq L$, $A \leqq L$, $A \trianglelefteq L$ for A is a subset, subalgebra, ideal of L respectively. If $A \subseteq L$, we denote by $\langle A \rangle$ the subspace spanned by A. If $A \leqq L$, we denote the normaliser and centraliser of A in L by $\mathcal{N}_L(A)$ and $\mathcal{C}_L(A)$ respectively. We denote the dimension of L by $\dim L$.

1. Homomorphs and Formations

Definition 1.1. (a) A class $\mathfrak{H}$ of finite-dimensional soluble Lie algebras is called a *homomorph* if $\mathfrak{H}$ contains, along with an algebra L, all epimorphic images of L.

(b) A homomorph $\mathfrak{H}$ is called a *formation* if $L/M, L/N \in \mathfrak{H}$ (where $M, N \trianglelefteq L$) implies $L/M \cap N \in \mathfrak{H}$.

Definition 1.2. Let $\mathfrak{H}$ be a homomorph and L a Lie algebra. We say that a subalgebra H of L is an $\mathfrak{H}$*-subalgebra* of L if

(i) $H \in \mathfrak{H}$, and

(ii) $H \leqq K \leqq L$, $K_0 \trianglelefteq K$, $K/K_0 \in \mathfrak{H}$ implies $H + K_0 = K_0$.

We denote the set of all $\mathfrak{H}$-subalgebras of L by $\mathfrak{H}(L)$ and write $H \in \mathfrak{H}(L)$ for H is an $\mathfrak{H}$-subalgebra of L.

Example 1.3. A subalgebra $H \leqq L$ is called a Cartan subalgebra of L if H is nilpotent and $\mathcal{N}_L(H) = H$. We denote by $\mathfrak{N}$ the formation of all nilpotent algebras. The $\mathfrak{N}$-subalgebras of L are precisely the Cartan subalgebras.

Proof. If $H \in \mathfrak{N}(L)$, then H is nilpotent. If $u \in \mathcal{N}_L(H)$, put $K = \langle u, H \rangle$. Then $K/H \in \mathfrak{N}$ and therefore $H + H = K$, which implies $u \in H$.

If H is a Cartan subalgebra, then $H \in \mathfrak{N}$. Suppose $H \leqq K \leqq L$, $K_0 \trianglelefteq K$ and $K/K_0 \in \mathfrak{N}$. By Barnes [2] Lemma 3, $H + K_0/K_0$ is a Cartan subalgebra of K/K_0. Therefore $H + K_0 = K$.

Lemma 1.4. *If $H\in\mathfrak{H}(L)$, then H is maximal in the set $\{K\leqq L\mid K\in\mathfrak{H}\}$. In particular, if $L\in\mathfrak{H}$, then $H=L$.*

Proof. If $H\leqq K\leqq L$ and $K\in\mathfrak{H}$, put $K_0=0$ in (ii).

Lemma 1.5. (a) *If $H\in\mathfrak{H}(L)$ and $H\leqq K\leqq L$, then $H\in\mathfrak{H}(K)$.*

(b) *If $H\in\mathfrak{H}(L)$ and $N\trianglelefteq L$, then $H+N/N\in\mathfrak{H}(L/N)$.*

Proof. Obvious.

Lemma 1.6. *Suppose $\mathfrak{H}$ is a homomorph which contains a non-zero algebra. Then the one-dimensional Lie algebra is in $\mathfrak{H}$, and $H\in\mathfrak{H}(L)$, $L\neq 0$ implies $H\neq 0$.*

Proof. Suppose $A\in\mathfrak{H}$, $A\neq 0$. Since A is soluble, there exists $A_1\lhd A$ such that $\dim A/A_1=1$. If $H\in\mathfrak{H}(L)$ and $L\neq 0$, it follows from Lemma 1.4 that $H\neq 0$.

Lemma 1.7. *Suppose $H\in\mathfrak{H}(L)$ and $0\neq H\leqq U\leqq L$. Then $\mathcal{N}_L(U)=U$.*

Proof. Take $k\in\mathcal{N}_L(U)$ and put $K=\langle k,U\rangle$. Then $K/U\in\mathfrak{H}$ and $H+U=K$. But $H\leqq U$ and therefore $k\in U$.

Lemma 1.8. *Suppose $N\trianglelefteq L$, $U/N\in\mathfrak{H}(L/N)$ and $H\in\mathfrak{H}(U)$. Then $H\in\mathfrak{H}(L)$.*

Proof. The result is trivial if $\dim L=0$. We use induction over $\dim L$. Suppose $H\leqq K\leqq L$, $K_0\trianglelefteq K$, $K/K_0\in\mathfrak{H}$. Then $K+N\geqq H+N=U$ and $U/N\in\mathfrak{H}(K+N/N)$. But $K+N/N\simeq K/N\cap K$ and therefore $U\cap K/N\cap K\in\mathfrak{H}(K/N\cap K)$.

Since $H\in\mathfrak{H}(U\cap K)$, if $K<L$, we have $H\in\mathfrak{H}(K)$ by induction and $H+K_0=K$. Suppose $K=L$. Since $L/K_0+N\in\mathfrak{H}$, $U/N+(K_0+N/N)=L/N$ and so $U+N+K_0=L$. Since $U\geqq N$, $U+K_0=L$. Therefore $U/K_0\cap U\simeq L/K_0\in\mathfrak{H}$. This implies $H+(K_0\cap U)=U$ and $H+K_0=U+K_0=L$.

Lemma 1.9. *Let $\mathfrak{H}$ be a homomorph and let M be a minimal ideal of L. Suppose $L/M\in\mathfrak{H}$, $L\notin\mathfrak{H}$ and $H\in\mathfrak{H}(L)$. Then H complements M in L.*

Proof. $H+M=L$ and $H\neq L$. Since M is an abelian ideal, $H\cap M\lhd H+M$. Therefore $H\cap M=0$.

Lemma 1.10. *Let $\mathfrak{F}$ be a formation and let M be a minimal ideal of L. Suppose $L/M\in\mathfrak{F}$, $L\notin\mathfrak{F}$ and that H complements M in L. Then $H\in\mathfrak{F}(L)$.*

Proof. $H\in\mathfrak{F}$ and is a maximal subalgebra of L. Thus we have only to prove that $N\trianglelefteq L$, $L/N\in\mathfrak{F}$ implies $H+N=L$. Since $\mathfrak{F}$ is a formation, $L/M\cap N\in\mathfrak{F}$. Since M is minimal and $L\notin\mathfrak{F}$, this implies $N\geqq M$ and $H+N\geqq H+M=L$.

Taking $\mathfrak{F}$ to be the homomorph of algebras of dimension at most one and L the 2-dimensional abelian algebra shows that the assumption that $\mathfrak{F}$ is a formation, not merely a homomorph, cannot be omitted from Lemma 1.10.

Lemma 1.11. *Suppose A is an abelian ideal of L, $L/A\in\mathfrak{H}$ and $U_1,U_2\in\mathfrak{H}(L)$. Then there exists $a\in A$ such that $U_2=U_1\exp d_a$, where d_a is the right inner derivation of L given by a.*

Proof. Since $d_a d_b=0$ for all $a,b\in A$, $\exp d_a$ is defined for all $a\in A$, and $\exp d_a\cdot\exp d_b=\exp d_{a+b}$.

Suppose $B\lhd L$, $0<B<A$. By induction, $U_2+B=(U_1+B)\exp d_a$ for some $a\in A$. By the remark above, we may suppose $U_1+B=U_2+B$. If $U_1+B=L$, then $U_1\cap A\lhd L$ and $U_1\cap A\neq 0$ since $L/B\simeq U_1/U_1\cap B$. Thus we could choose

$B \leqq U_1 \cap A$ in this case, and $U_1 + B = U_2 + B$ then gives $U_1 = U_2$. Thus we may suppose $U_1 + B \neq L$. By induction, there exists $b \in B$ such that $U_2 = U_1 \exp d_b$.

We have now only to consider the case in which A is minimal and U_1, U_2 are complements of A in L. By Barnes [1] Theorem 4, the result holds if $\mathscr{C}_L(A) = A$. Suppose $\mathscr{C}_L(A) > A$. Then there exists a minimal ideal B of L, $B \leqq U_1$. Since $U_i + B/B \in \mathfrak{H}(L/B)$, by induction there exists $a \in A$ such that $U_2 + B/B = (U_1/B) \exp d_{\bar{a}}$, where $\bar{a} = a + B \in A + B/B$. Thus $U_2 + B = U_1 \exp d_a \in \mathfrak{H}$ which implies $U_2 = U_2 + B$.

For any subalgebra A of L, we denote by $\mathscr{I}(L, A)$ the group of automorphisms of L generated by the exponentials $\exp d_a$ for those $a \in A$ for which $\exp d_a$ is meaningful. For $\exp d_a$ to be meaningful, we must have d_a nilpotent and, if *char* $F = p \neq 0$, $d_a^p = 0$.

Lemma 1.12. *Let $\mathfrak{H}$ be a homomorph and let A be a minimal ideal of L. Suppose $L/A \in \mathfrak{H}$, $L \notin \mathfrak{H}$ and that $\mathfrak{H}(L) \neq \emptyset$. Then $\mathfrak{H}(L)$ is the set of all complements to A in L, $H^1(L/A, A) = 0$ and all complements to A in L are conjugate under $\mathscr{I}(L, A)$.*

Proof. If $H \in \mathfrak{H}(L)$, then H is a complement to A in L by Lemma 1.9. If U complements A in L, then there is an automorphism α of L such that $H^\alpha = U$. Therefore $U \in \mathfrak{H}(L)$ and by Lemma 1.11, H and U are conjugate under $\mathscr{I}(L, A)$. That all complements to A in the split extension L of A by L/A are conjugate under $\mathscr{I}(L, A)$ is equivalent to $H^1(L/A, A) = 0$.

Theorem 1.13. *Let $\mathfrak{H}$ be a homomorph, L a soluble Lie algebra and $H_1, H_2 \in \mathfrak{H}(L)$. If char $F = p \neq 0$, suppose further that the intersection L_ω of all terms of the descending central series of L satisfies the $(p-1)^{\text{st}}$ Engel condition. Then H_1 and H_2 are conjugate under $\mathscr{I}(L, L_\omega)$.*

Proof. If L is nilpotent, then by Lemma 1.7, we have either $H_1 = H_2 = 0$ or $H_1 = H_2 = L$. We may therefore suppose $L_\omega \neq 0$. We use induction over $\dim L$. Let $A \leqq L_\omega$ be a minimal ideal of L. Then $H_1 + A/A$ and $H_2 + A/A$ are conjugate under $\mathscr{I}(L/A, (L/A)_\omega)$. But as in Barnes [2] Lemma 5, each element of $\mathscr{I}(L/A, (L/A)_\omega)$ is induced by an element of $\mathscr{I}(L, L_\omega)$. Hence we may assume $H_1 + A = H_2 + A$. If $H_1 + A < L$, then by induction, H_1 and H_2 are conjugate under $\mathscr{I}(H_1 + A, (H_1 + A)_\omega)$ and so under $\mathscr{I}(L, L_\omega)$. If $H_1 + A = L$, then H_1 and H_2 are conjugate under $\mathscr{I}(L, A)$ by Lemma 1.11.

As was shown in Barnes [2] Example 2, the condition in the case of characteristic p, that L_ω satisfies the $(p-1)^{\text{st}}$ Engel condition cannot be omitted.

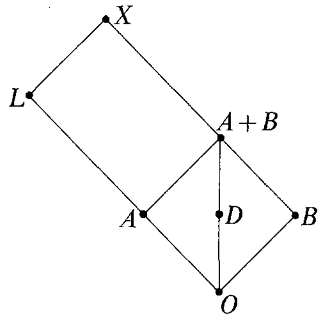
Lemma 1.14. *Let $\mathfrak{H}$ be a homomorph which is not a formation. Then there exists a Lie algebra $L \notin \mathfrak{H}$, with distinct minimal ideals M, N such that L/M, $L/N \in \mathfrak{H}$.*

Proof. There exists an algebra L with ideals M_1, N_1 such that $L/M_1, L/N_1 \in \mathfrak{H}$ but $L/M_1 \cap N_1 \notin \mathfrak{H}$. Choose such an algebra L of least possible dimension. Then $M_1 \cap N_1 = 0$. Take M a minimal ideal of L contained in M_1. Then L/M_1, $L/M + N_1 \in \mathfrak{H}$ and $M_1 \cap (M + N_1) = M$. By the minimality of L, $L/M \in \mathfrak{H}$. Similarly, if N is a minimal ideal of L contained in N_1, then $L/N \in \mathfrak{H}$.

Definition 1.15. A homomorph $\mathfrak{H}$ is said to be *split* if $L \in \mathfrak{H}$, and A an abelian ideal of L, implies that the split extension of A by L/A also belongs to $\mathfrak{H}$.

Lemma 1.16. *Let $\mathfrak{F}$ be a formation. Then $\mathfrak{F}$ is split.*

Proof. Suppose $L \in \mathfrak{F}$ and that A is an abelian ideal of L. Take $B \simeq A$ as L-module and form the split extension X of B by L.



Since $A+B$ is the direct sum of two isomorphic X-modules, there exists a diagonal $D \lhd X$. We have $L+D=L+A+B=X$, $L \cap D=0$ and $X/D \simeq X/B \simeq L \in \mathfrak{F}$. Therefore $X = X/B \cap D \in \mathfrak{F}$ and so $X/A \in \mathfrak{F}$. But X/A is the split extension of $A+B/A\,(\simeq A)$ by L/A.

2. Functors

Definition 2.1. (a) We say that f is a *functor* defined on the homomorph Ω if, for each $L \in \Omega$, $f(L)$ is a non-empty set of subalgebras of L and, for every epimorphism $\varphi: L \to L^{\varphi}$,

$$f(L^{\varphi}) = \{U^{\varphi} \mid U \in f(L)\}.$$

(b) We say that the functor f is *regular* if $L \in f(L)$ implies $f(L)=\{L\}$.

(c) We say that the functor f is *inclusive* if $U \in f(L)$, $U \leqq M \leqq L$ implies $M \in \Omega$ and $U \in f(M)$.

We write "*functor*" for "*regular inclusive functor*" as no other functors will be considered.

Definition 2.2. We define the *eigenclass* of a functor f to be the class $\{L \in \Omega \mid L \in f(L)\}$. It is clearly a homomorph.

Example 2.3. Let $\mathfrak{H}$ be any homomorph, and let Ω be the class of all algebras L for which $\mathfrak{H}(L) \neq \emptyset$. Then Ω is a homomorph, and we have a functor $f_{\mathfrak{H}}$ defined on Ω by $f_{\mathfrak{H}}(L) = \mathfrak{H}(L)$. The eigenclass of $f_{\mathfrak{H}}$ is $\mathfrak{H}$. We shall prove that every functor is given in this way by its eigenclass.

In the following three lemmas, f is a functor defined on Ω and $\mathfrak{H}$ is the eigenclass of f.

Lemma 2.4. $f(L) \subseteq \mathfrak{H}(L)$ *for all* $L \in \Omega$.

Proof. Suppose $U \in f(L)$, $U \leqq K \leqq L$, $K_0 \lhd K$ and $K/K_0 \in \mathfrak{H}$. Then $U \in f(K)$ and therefore $U+K_0/K_0 \in f(K/K_0) = \{K/K_0\}$. Thus $U+K_0=K$.

Lemma 2.5. *Suppose $L\in\Omega$, A is an abelian ideal of L and $L/A\in\mathfrak{H}$. Then $f(L)=\mathfrak{H}(L)$.*

Proof. There exists $U\in f(L)\subseteq\mathfrak{H}(L)$. Suppose $V\in\mathfrak{H}(L)$. Then U and V are conjugate under $\mathscr{I}(L,A)$ by Lemma 1.11. Therefore $V\in f(L)$.

Lemma 2.6. *Suppose $L\in\Omega$, A is an abelian ideal of L, $U/A\in f(L/A)$ and $H\in f(U)$. Then $H\in f(L)$.*

Proof. Since f is a functor, there exists $V\in f(L)$ such that $V+A=U$. Then $V, H\in f(U)\subseteq\mathfrak{H}(U)$. By Lemma 1.11, V and H are conjugate under $\mathscr{I}(U,A)$ and so under $\mathscr{I}(L,A)$. Therefore $H\in f(L)$.

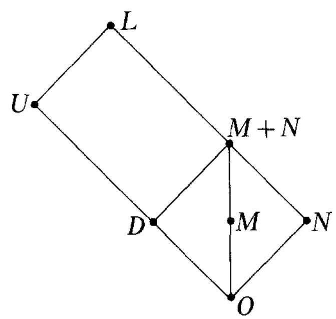
Theorem 2.7. *Let $\mathfrak{H}$ be the eigenclass of the functor f defined on Ω. Then $f(L)=\mathfrak{H}(L)$ for all $L\in\Omega$.*

Proof. Suppose $H\in\mathfrak{H}(L)$. We use induction over $\dim L$ to prove $H\in f(L)$. Let A be a minimal ideal of L. Then $H+A/A\in f(L/A)$. The result holds by Lemma 2.5 if $H+A=L$. Suppose $H+A<L$. By induction, $H\in f(H+A)$ and therefore $H\in f(L)$ by Lemma 2.6.

We now show that the functors which arise as in 2.3 from a *formation* are characterised by having an eigenclass which is split.

Theorem 2.8. *Let f be a functor defined on a formation Ω and let $\mathfrak{H}$ be the eigenclass of f. Suppose $\mathfrak{H}$ is split. Then $\mathfrak{H}$ is a formation.*

Proof. Suppose $\mathfrak{H}$ is not a formation. By Lemma 1.14, there exists an algebra $L\notin\mathfrak{H}$ with distinct minimal ideals M, N such that $L/M, L/N\in\mathfrak{H}$. Since Ω is a formation and $\mathfrak{H}\subseteq\Omega$, we have $L\in\Omega$.



Take $U\in f(L)$. Then $U+M=U+N=L$ and $U\neq L$. Put $D=U\cap(M+N)$. Then $D\simeq M\simeq N$. Since L/D is the split extension of $M+N/D$ ($\simeq M+N/M$) by U/D ($\simeq L/M+N$) and $L/M\in\mathfrak{H}$ which is split, $L/D\in\mathfrak{H}$. But this contradicts $U/D\in f(L/D)$.

3. Saturation

Definition 3.1. A homomorph $\mathfrak{H}$ is said to be *saturated* if all soluble Lie algebras have $\mathfrak{H}$-subalgebras, that is, if the domain of definition of the functor $f_{\mathfrak{H}}$ is the formation $\mathfrak{S}$ of all soluble Lie algebras. We note that a saturated homomorph cannot be empty.

Theorem 3.2. *Let $\mathfrak{H}$ be a non-empty homomorph. A necessary and sufficient condition for $\mathfrak{H}$ to be saturated is that $L \notin \mathfrak{H}$, M a minimal ideal of L and $L/M \in \mathfrak{H}$ implies $\mathfrak{H}(L) \neq \emptyset$.*

Proof. The condition is trivially necessary. Suppose $\mathfrak{H}$ satisfies the condition. We use induction over $\dim L$ to prove for all L that $\mathfrak{H}(L) \neq \emptyset$. Let M be a minimal ideal of L. Then there exists $U \leqq L$ such that $U \geqq M$, $U/M \in \mathfrak{H}(L/M)$.

If $U < L$, then by induction, there exists $H \in \mathfrak{H}(U)$ and by Lemma 1.8. $H \in \mathfrak{H}(L)$. If $U = L$, then $L/M \in \mathfrak{H}$. Either $L \in \mathfrak{H}$ and $L \in \mathfrak{H}(L)$ or $L \notin \mathfrak{H}$ and $\mathfrak{H}(L) \neq \emptyset$ by hypothesis.

Corollary 3.3. *Let $\mathfrak{F}$ be a non-empty formation. A necessary and sufficient condition for $\mathfrak{F}$ to be saturated is that $L \notin \mathfrak{F}$, M a minimal ideal of L and $L/M \in \mathfrak{F}$ implies that there exists a complement to M in L.*

Proof. Suppose $L \notin \mathfrak{F}$, M is a minimal ideal of L and $L/M \in \mathfrak{F}$. By Lemmas 1.9, 1.10, $U \in \mathfrak{F}(L)$ if and only if U complements M in L.

The Frattini subalgebra $\Phi(L)$ of a Lie algebra L is the intersection of the maximal subalgebras of L.

Lemma 3.4. *Let L be a soluble Lie algebra. Then $\Phi(L)$ is an ideal of L.*

Proof. We use induction over $\dim L$. The result is trivial for $\dim L = 1$. Let A be a minimal ideal of L. Denote by Φ_A the intersection of the maximal subalgebras of L which contain A. Then $\Phi_A/A = \Phi(L/A)$ and $\Phi_A \lhd L$ by induction.

If $A \leqq \Phi(L)$, then $\Phi(L) = \Phi_A$ is an ideal. Suppose $A \nleqq \Phi(L)$. If M is a maximal subalgebra and $M \ngeqq A$, then M is a complement to A in L. If $\mathscr{C}_L(A) > A$, then $0 < M \cap \mathscr{C}_L(A) \lhd L$ and every maximal subalgebra M of L contains some minimal ideal B of L. In this case, $\Phi(L) = \bigcap_B \Phi_B$ and is therefore an ideal.

Suppose $\mathscr{C}_L(A) = A$. We prove $\Phi(L) = 0$. Let M be a maximal subalgebra not containing A and suppose $m \in M$, $m \neq 0$. We prove $m \notin \Phi(L)$. Since $m \notin \mathscr{C}_L(A)$, there exists $a \in A$ such that $ma \neq 0$. Since $d_a^2 = 0$, $\exp d_a = 1 + d_a$ is an automorphism of L. Put $M_1 = M \exp d_a$. If $m \in M_1$, then $m = m'(1 + d_a) = m' + m'a$ for some $m' \in M$. But this implies $m'a \in M \cap A = 0$, $m = m'$ and $ma = 0$ contrary to the choice of a.

Lemma 3.5. *Let $\mathfrak{H}$ be a saturated homomorph. Then*

(i) *$L, M \in \mathfrak{H}$ implies $L \oplus M \in \mathfrak{H}$.*

(ii) *$L/\Phi(L) \in \mathfrak{H}$ implies $L \in \mathfrak{H}$.*

Proof. (i) Let $X = L \oplus M$ be a minimal counterexample. Since $\mathfrak{H}$ is saturated, there exists $H \in \mathfrak{H}(X)$. We have $H \neq X$, $H + L = H + M = X$. Suppose $H \cap L = A \neq 0$. Then $H, M \leqq \mathscr{N}_X(A)$ and therefore $A \trianglelefteq X$. Since $X/A \simeq (L/A) \oplus M \in \mathfrak{H}$, $H + A = X$ contrary to $X > H \geqq A$. Therefore $H \cap L = H \cap M = 0$ and $L \simeq H \simeq M$. Let A be a minimal ideal of L. Then $H + A = X$ and $X/A \simeq H \simeq X/L$. Thus $L = A$ and X is abelian. But $\mathscr{N}_X(H) = H < X$.

(ii) We use induction over $\dim L$. If $\Phi(L) = 0$, then the result is trivial. Suppose $\Phi(L) \neq 0$. Since $\Phi(L)$ is an ideal, there exists a minimal ideal A of L,

$A \leqq \Phi(L)$. We have $\Phi(L/A) = \Phi(L)/A$ and by induction, $L/A \in \mathfrak{H}$. If $L \notin \mathfrak{H}$, take $H \in \mathfrak{H}(L)$. By Lemma 1.9, H complements A in L contrary to $A \leqq \Phi(L)$.

Lemma 3.6. *Suppose $\mathfrak{F}$ is a non-empty formation and $L/\Phi(L) \in \mathfrak{F}$ implies $L \in \mathfrak{F}$. Then $\mathfrak{F}$ is saturated.*

Proof. We have to show that $L \notin \mathfrak{F}$, A a minimal ideal of L and $L/A \in \mathfrak{F}$ implies that there exists a complement U of A in L. If $A \leqq \Phi(L)$, then $L/\Phi(L) \in \mathfrak{F}$. Therefore $A \nleqq \Phi(L)$, and there exists a maximal subalgebra U of L not containing A. Then $U + A = L$ and $U \cap A = 0$.

We now give some examples of saturated formations.

Lemma 3.7. *The following are saturated formations:*

(i) *The class $\mathfrak{O}$ containing the trivial algebra* 0 *only.*

(ii) *The class $\mathfrak{N}$ of all nilpotent algebras.*

(iii) *The class $\mathfrak{U}$ of all supersoluble algebras.*

(iv) *The class $\mathfrak{S}$ of all soluble algebras.*

Proof. $\mathfrak{O}$ and $\mathfrak{S}$ are trivially saturated formations. $\mathfrak{N}$ and $\mathfrak{U}$ are easily seen to be formations. That they are saturated follows from Lemma 3.6 and Barnes [1], Theorems 5, 6.

4. Construction of Saturated Homomorphs

For the discussion of the analogue of Gaschütz's concept of local definition of saturated formations ([4] § 3), we need some properties of the Fitting subalgebra.

Definition 4.1. The *Fitting subalgebra* $\mathscr{F}(L)$ of a Lie algebra L is the (unique) maximal nilpotent ideal of L.

Definition 4.2. Let U, V be ideals of L, $V \leqq U$. Then U/V is an L-module. The *centraliser* $\mathscr{C}_L(U/V)$ of U/V in L is defined to be the kernel of the representation of L on U/V, that is

$$\mathscr{C}_L(U/V) = \{x \in L \mid x\,U \subseteq V\}.$$

Lemma 4.3. *Let L be a (not necessarily soluble) Lie algebra. Then $\mathscr{F}(L)$ is the intersection of the centralisers of the factors of a principal series of L.*

Proof. By the Jordan-Hölder theorem, the intersection D of the centralisers is independent of the choice of principal series. It is trivially an ideal of L. If U/V is a principal factor of L and $D \geqq U$, then U/V is in the centre of L/V and so of D/V. Therefore D is nilpotent.

Suppose N is a nilpotent ideal of L. Take a central series of N and refine it to a principal series of L. Let U/V be a factor of this series. If $N \geqq U$, then U/V is central in N/V and $\mathscr{C}_L(U/V) \geqq N$. If $N \ngeqq U$, then $V \geqq N$ and $\mathscr{C}_L(U/V) \geqq N$. Therefore $N \leqq D$.

Lemma 4.4. *Let A be a minimal ideal of the (not necessarily soluble) Lie algebra L, $A \leqq \mathscr{F}(L)$. Suppose $\mathscr{F}(L/A) > \mathscr{F}(L)/A$. Then $H^n(L/A, A) = H_n(L/A, A) = 0$ for all n.*

Proof. Let $U/A = \mathscr{F}(L/A)$. Then $U \trianglelefteq L$, U/A is nilpotent but U is not nilpotent. Therefore $UA \neq 0$ and so $UA = A$. By Barnes [1] Theorem 1, $H^\beta(U/A, A) = H_\beta(U/A, A) = 0$ for all β. Therefore $H^\alpha(L/U, H^\beta(U/A, A)) = H_\alpha(L/U, H_\beta(U/A, A)) = 0$ for all α, β and the result follows by the Hochschild-Serre spectral sequences.

Definition 4.5. Let $\mathfrak{F}$ be a formation. Put

$$\mathfrak{L}_{\mathfrak{F}} = \{L \in \mathfrak{S} \mid L/\mathscr{C}_L(U/V) \in \mathfrak{F} \textit{ for every principal factor } U/V \textit{ of } L\}.$$

Then $\mathfrak{L}_{\mathfrak{F}}$ is clearly a formation. We call it the formation *locally defined by* $\mathfrak{F}$.

If $\mathfrak{F} = \emptyset$ and $L \in \mathfrak{L}_{\mathfrak{F}}$, then L has no principal factors, that is $L = 0$. Thus $\mathfrak{L}_\emptyset = \mathfrak{D}$. Clearly $\mathfrak{L}_{\mathfrak{D}} = \mathfrak{N}$ and $\mathfrak{L}_{\mathfrak{S}} = \mathfrak{S}$.

Theorem 4.6. *For any formation $\mathfrak{F}$, $\mathfrak{L}_{\mathfrak{F}}$ is saturated. If A is a minimal ideal of L, $L/A \in \mathfrak{L}_{\mathfrak{F}}$ but $L \notin \mathfrak{L}_{\mathfrak{F}}$, then $H^n(L/A, A) = H_n(L/A, A) = 0$ for all n.*

Proof. Trivially $\mathfrak{L}_{\mathfrak{F}} \neq \emptyset$. By Corollary 3.3, it is sufficient to prove the second assertion. Put

$$N = \cap \{\mathscr{C}_L(U/V) \mid U/V \textit{ a principal factor of } L, V \geqq A\}.$$

Since $\mathfrak{F}$ is a formation and $L/\mathscr{C}_L(U/V) \in \mathfrak{F}$ for all principal factors U/V of L with $V \geqq A$, $L/N \in \mathfrak{F}$. If $\mathscr{F}(L/A) = \mathscr{F}(L)/A$, then $\mathscr{C}_L(A) \geqq N$ and $L/\mathscr{C}_L(A) \in \mathfrak{F}$ contrary to $L \notin \mathfrak{L}_{\mathfrak{F}}$. Therefore $\mathscr{F}(L/A) > \mathscr{F}(L)/A$ and the result follows by Lemma 4.4.

In view of Theorem 4.6, one might reasonably ask if every saturated formation is locally definable. We shall see later that this is (trivially) true if the ground field is algebraically closed of characteristic 0. It is not true in general.

Lemma 4.7. *Suppose the ground field F is not algebraically closed. Then the saturated formation $\mathfrak{U}$ of supersoluble algebras is not locally definable.*

Proof. If $\mathfrak{U} \subseteq \mathfrak{L}_{\mathfrak{F}}$, then the one-dimensional algebra is in $\mathfrak{F}$. Since F is not algebraically closed, there exists a monic irreducible polynomial $m(x)$ over F of degree $r > 1$. Let A be a vector space of dimension r and let $\alpha: A \to A$ be a linear transformation with characteristic polynomial $m(x)$. Put $L = \langle e, A \rangle$, $ab = 0$, $ea = \alpha(a)$ for all $a, b \in A$. This makes L a soluble Lie algebra with A a minimal ideal and $L/\mathscr{C}_L(A)$ one-dimensional. Since $L \in \mathfrak{L}_{\mathfrak{F}}$ but $L \notin \mathfrak{U}$, the formation $\mathfrak{U}$ is not locally definable.

Definition 4.8. A Lie algebra $P \neq 0$ is called *primitive* if it has a minimal ideal A such that $\mathscr{C}_P(A) = A$.

Definition 4.9. Let $\mathfrak{H}$ be a homomorph. Put

$$\mathfrak{P}(\mathfrak{H}) = \{L \in \mathfrak{S} \mid M \trianglelefteq L, L/M \textit{ primitive implies } L/M \in \mathfrak{H}\}.$$

Then $\mathfrak{P}(\mathfrak{H})$ is trivially a non-empty homomorph. We call it the homomorph *primitively defined by* $\mathfrak{H}$. Clearly, we may replace $\mathfrak{H}$ in this definition by its primitive residue $\mathscr{R}(\mathfrak{H})$, that is, the set of isomorphism classes of primitive algebras in $\mathfrak{H}$.

Lemma 4.10. *For any homomorph* $\mathfrak{H}$, $\mathfrak{P}(\mathfrak{H})$ *is saturated.*

Proof. Put $\mathfrak{K}=\mathfrak{P}(\mathfrak{H})$. We prove by induction over $\dim L$ that every Lie algebra L has $\mathfrak{K}$-subalgebras. We may suppose $L\notin\mathfrak{K}$. Let A be a minimal ideal of L. Then there exists $U\geqq A$, $U/A\in\mathfrak{K}(L/A)$. If $U<L$, then there exists $K\in\mathfrak{K}(U)$ and by Lemma 1.8 $K\in\mathfrak{K}(L)$. Hence we may suppose $L/A\in\mathfrak{K}$ for every minimal ideal A of L. If for every minimal ideal A, $\mathscr{C}_L(A)>A$, then every primitive factor algebra of L is a primitive factor algebra of some L/A and so is in $\mathfrak{H}$. Since $L\notin\mathfrak{K}$, there exists a minimal ideal A of L such that $\mathscr{C}_L(A)=A$. By Barnes [1] Theorem 4, there exists a complement U to A in L. Since U is a maximal subalgebra and A is the *unique* minimal ideal of L, it follows that $U\in\mathfrak{K}(L)$.

Theorem 4.11. *Let* $\mathfrak{H}$ *be a homomorph. Then* $\mathfrak{P}(\mathfrak{H})$ *is the smallest saturated homomorph containing* $\mathfrak{H}$.

Proof. Clearly $\mathfrak{H}\subseteq\mathfrak{P}(\mathfrak{H})$. If $\mathfrak{H}_1\subseteq\mathfrak{H}_2$, then $\mathfrak{P}(\mathfrak{H}_1)\subseteq\mathfrak{P}(\mathfrak{H}_2)$. It is therefore sufficient to prove that if $\mathfrak{H}$ is saturated, then $\mathfrak{P}(\mathfrak{H})=\mathfrak{H}$. Suppose $\mathfrak{H}$ is saturated and $\mathfrak{P}(\mathfrak{H})\neq\mathfrak{H}$. Take $L\in\mathfrak{P}(\mathfrak{H})$, $L\notin\mathfrak{H}$ of least possible dimension. Let A be a minimal ideal of L. Then $L/A\in\mathfrak{H}$. Take $H\in\mathfrak{H}(L)$. By Lemma 1.9, H is a complement to A in L. If $\mathscr{C}_L(A)>A$, then $B=H\cap\mathscr{C}_L(A)\neq 0$ and is an ideal of L. This implies $L/B\in\mathfrak{H}$ and $H=L$. Therefore $\mathscr{C}_L(A)=A$, L is primitive and so $L\in\mathfrak{H}$ contrary to assumption.

Corollary 4.12. *The saturated homomorphs, partially ordered by inclusion, form a complete distributive lattice.*

Proof. For any set $\{\mathfrak{H}_\alpha\,|\,\alpha\in I\}$ of homomorphs, indexed by some index set I, $\bigcap_\alpha\mathfrak{H}_\alpha$ and $\bigcup_\alpha\mathfrak{H}_\alpha$ are again homomorphs, and

$$\bigcap_\alpha\mathscr{R}(\mathfrak{H}_\alpha)=\mathscr{R}\Big(\bigcap_\alpha\mathfrak{H}_\alpha\Big),\qquad \bigcup_\alpha\mathscr{R}(\mathfrak{H}_\alpha)=\mathscr{R}\Big(\bigcup_\alpha\mathfrak{H}_\alpha\Big).$$

Thus $\{\mathscr{R}(\mathfrak{H})\,|\,\mathfrak{H}$ *homomorph*$\}$ is a complete sublattice of the lattice of subsets of the set of isomorphism classes of primitive algebras. But each $\mathscr{R}(\mathfrak{H})$ is $\mathscr{R}(\mathfrak{K})$ for a unique saturated homomorph $\mathfrak{K}$.

It follows that to determine the saturated homomorphs for some fixed ground field F, it is sufficient to determine the primitive algebras over F and their primitive factor algebras.

Lemma 4.13. *The formations* $\mathfrak{N}$, $\mathfrak{U}$ *of nilpotent and supersoluble algebras are determined by the primitive residues*

$$\mathscr{R}(\mathfrak{N})=\{\langle e\rangle\}$$

$$\mathscr{R}(\mathfrak{U})=\{\langle e\rangle,\langle e,a\,|\,ea=a\rangle\}.$$

If $\mathfrak{H}$ is a saturated homomorph and $\mathfrak{H}$ contains a non-zero algebra, then $\mathfrak{H} \supseteq \mathfrak{N}$. If $\mathfrak{U} \supseteq \mathfrak{H} \supset \mathfrak{N}$, then $\mathfrak{H} = \mathfrak{U}$.

Proof. Since $\langle e \rangle$ (the one-dimensional algebra) is the only nilpotent primitive algebra, $\mathscr{R}(\mathfrak{N}) = \{\langle e \rangle\}$. If $\mathfrak{H}$ is saturated and contains a non-zero algebra, then $\langle e \rangle \in \mathfrak{H}$ which implies $\mathfrak{N} \subseteq \mathfrak{H}$. If $\mathfrak{H} \supset \mathfrak{N}$, then there exists a non-nilpotent primitive algebra $P \in \mathfrak{H}$. Let A be the minimal ideal of P. If P is supersoluble, then $\dim A = 1$ and $A = \langle a \rangle$ for some element a. Since P/A is faithfully represented on A, $\dim P/A = 1$ and it follows that there exists $e \in P$ such that $P = \langle e, a \rangle$ and $ea = a$.

Corollary 4.14. *The only saturated homomorphs for an algebraically closed ground field of characteristic* 0 *are* $\mathfrak{O}$, $\mathfrak{N}$, $\mathfrak{S}$.

Proof. For algebraically closed fields of characteristic 0, $\mathfrak{U} = \mathfrak{S}$.

Corollary 4.15[1]**.** *Let F be an algebraically closed field of characteristic* 0. *Let f be a regular inclusive functor defined on all (not necessarily soluble) finite-dimensional Lie algebras over F. Suppose that for all L, the subalgebras $U \in f(L)$ are soluble. Then f is the zero, Cartan or Borel functor.*

Proof. If $U \in f(L)$, then there exists a Borel subalgebra $B \geqq U$. U is a Cartan subalgebra of L if and only if U is a Cartan subalgebra of B. Also, we have $U \in f(B)$. Thus we need only consider the restriction of f to the formation of soluble Lie algebras, and the result follows by Corollary 4.14.

For algebraically closed fields of characteristic 0, the three concepts, saturated homomorph, saturated formation and locally definable formation coincide. This is not true for arbitrary fields. As we have already seen, if the field is not algebraically closed, then the saturated formation $\mathfrak{U}$ of supersoluble algebras is not locally definable. We now show that there exist fields for which not all saturated homomorphs are formations.

Lemma 4.16. *Let F be any field such that there exists an irreducible polynomial $f(x)$ over F whose degree is greater than* 1 *and not divisible by the characteristic of F. Then there exists a saturated homomorph of Lie algebras over F which is not a formation.*

Remark. Among the fields considered are all finite fields and all fields of characteristic 0 which are not algebraically closed.

Proof. There exists a polynomial $m(x)$ over F of degree $r > 1$, the sum of whose roots is not 0 since, if the sum of the roots of $f(x)$ is 0, we may take $m(x) = f(x-1)$. Take A a vector space of dimension r and α: $A \to A$ a linear transformation whose characteristic polynomial is $m(x)$. Put $P = \langle u, A \rangle$, $ua = \alpha(a)$, $ab = 0$ for all $a, b \in A$. Then P is a primitive algebra with minimal ideal A. The only primitive proper factor algebra of P is the one-dimensional algebra. Let $\mathfrak{H}$ be the saturated homomorph whose primitive algebras are P and the one-dimensional algebra. We prove that $\mathfrak{H}$ is not split.

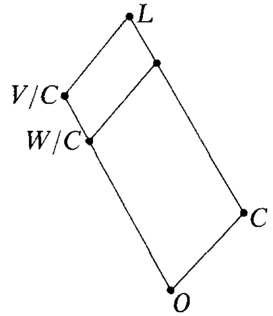
[1] Communicated to the authors by O. H. Kegel.

Let E be the free nilpotent Lie algebra of class r on r generators. E may be regarded as a Lie subalgebra of the factor algebra by the elements of degree $>r$ of the tensor algebra on A. Let E_i be the component of terms of degree i. The linear transformation α of $E_1(\simeq A)$ induces a linear transformation α_i of E_i, defined inductively by

$$\alpha_i(u\otimes v)=\alpha(u)\otimes v+u\otimes\alpha_{i-1}(v)$$

for $u\in E_1$ and $v\in E_{i-1}$. Let $\lambda_1,\ldots,\lambda_r$ be the roots of $m(x)$, not necessarily distinct. Then $\lambda_1+\cdots+\lambda_r\in F$ is an eigenvalue of α_r. We form a Lie algebra $U=\langle u,E\rangle$, defining $uv=\alpha_i(v)$ for $v\in E_i$. There exists $e\in E_r$, $e\neq 0$ such that $\langle u,e\rangle$ is primitive. Clearly $\langle u,e\rangle\notin\mathfrak{H}$. We form the factor algebra $V=U/K$ by an ideal K maximal with respect to the requirement that U/K has a principal factor C whose split extension by $\langle u\rangle$ is a primitive algebra not in $\mathfrak{H}$. Then $K<E_2$ and C is the unique minimal ideal of V. Put $W=E/K\lhd V$. Since W is nilpotent, the only primitive factor algebras of V are V/W and U/E_2. Thus $V\in\mathfrak{H}$.

Let L be the split extension of C by V/C. The centraliser of C in L is W/C. Thus $L/(W/C)$ is the split extension of C by $\langle u\rangle$ and so is a primitive algebra not in $\mathfrak{H}$. Thus $L\notin\mathfrak{H}$. But an extension V of C by V/C is in $\mathfrak{H}$.



Finally, we show that, in the study of a regular inclusive functor, we may always suppose that the homomorph of definition of the functor is the homomorph of all soluble Lie algebras.

Lemma 4.17. *Suppose $H\in\mathfrak{H}(L)$. Then $H\in(\mathfrak{P}(\mathfrak{H}))(L)$.*

Proof. Suppose $H\leqq K\leqq L$, $K_0\lhd K$ and $K/K_0\in\mathfrak{P}(\mathfrak{H})$. We have to prove $H+K_0=K$. We use induction over $\dim K/K_0$. Without loss of generality, we may suppose $K=L$, $K_0=0$. Let M be a minimal ideal of L. Then by induction, $H+M=L$. Suppose $H\neq L$. Then H is a complement to M in L. Put $D=H\cap\mathscr{C}_L(M)$. Then L/D is primitive and therefore $L/D\in\mathfrak{H}$, which, by assumption, implies $H+D=L$. But $D\leqq H<L$. Thus $H\neq L$ is impossible.

Corollary 4.18. *Suppose f is a regular inclusive functor defined on the homomorph Ω. Then there exists a functor f^* defined on all soluble Lie algebras, such that f^* coincides with f on Ω.*

Proof. The result is an immediate consequence of Theorem 2.7 and Lemma 4.17.

References

1. Barnes, D. W.: On the cohomology of soluble Lie algebras. Math. Zeitschr. **101**, 343–349 (1967).
2. – On Cartan subalgebras of Lie algebras. Math. Zeitschr. **101**, 350–355 (1967).
3. –, and O. H. Kegel: Gaschütz functors on finite soluble groups. Math. Zeitschr. **94**, 134–142 (1966).
4. Gaschütz, W.: Zur Theorie der endlichen auflösbaren Gruppen. Math. Zeitschr. **80**, 300–305 (1963).
5. –, u. U. Lubeseder: Kennzeichnung gesättigter Formationen. Math. Zeitschr. **82**, 198–199 (1963).
6. Schunck, H.: $\mathfrak{H}$-Untergruppen in endlichen auflösbaren Gruppen. Math. Zeitschr. **97**, 326–330 (1967).
7. Sudbrock, W.: Sylowfunktionen in endlichen Gruppen. Rendiconti Sem. Mat. Padova **36**, 158–184 (1966).

Dr. D. W. Barnes
Department of Pure Mathematics
University of Sydney
Sydney, N.S.W., 2006 (Australia)

(Received November 21, 1967)